

*** Linear Algebra ***

- In General there are 2 - types OF Linear equation.

Linear equation :-

$$Ax = B \text{ [Non-Homogenous]}$$

$$Ax = 0 \text{ [Homogenous]}$$

Here, A = Coefficient matrix.

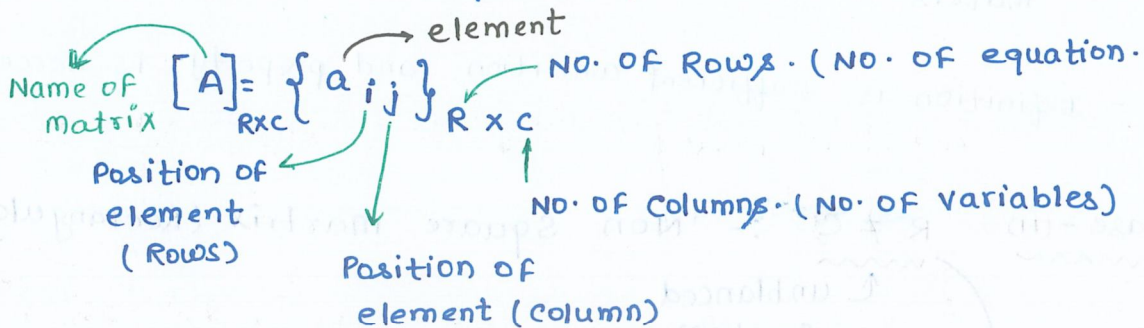
x = unknown matrix (Variable)

B = known matrix

Matrix :-

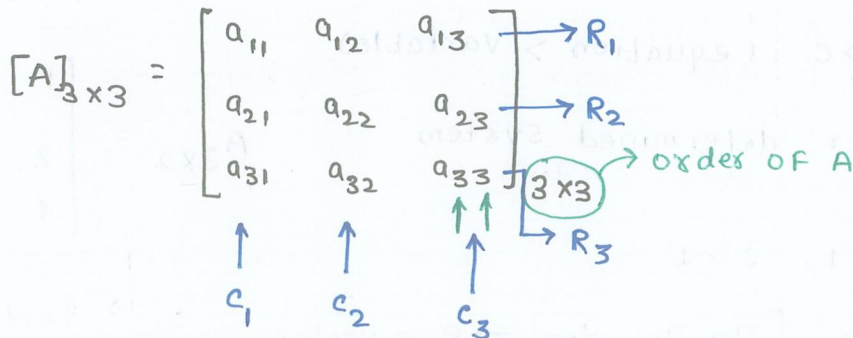
- matrix is the collection OF Number (Real or complex).

in fixed no. OF Rows/Columns.



$(R \times C)$:- NO. of elements = Area OF matrix
(Altogether dimensions).

ex:-



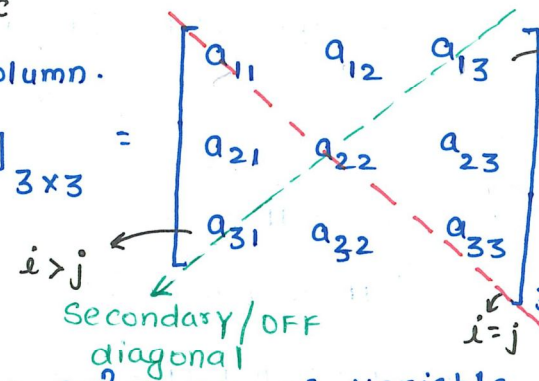
Types of matrix A/c to its dimensions (R, C)

Definition is sufficient condition.
 Definition & Property

Case-(i)

$R = C$ (NO. OF eqⁿ = no. of variable)
 Row = Column.

$$[A]_{3 \times 3} =$$



$i < j$
 Square matrix.
 ($R = C$)

$i > j$
 Secondary/OFF diagonal

$i = j$
 main/ON/Leading/Primary diagonal

- If no. of eqⁿ = no. of variable is called Balanced System / Determined system.

NOTE :- A Square matrix is very useful in Linear Algebra because diagonal, determinant, Adjoint, Inverse, unique solution, characteristic eqⁿ. are only valid for square matrix.

- Definition is sufficient condition and property is necessary condⁿ?

Case-(ii)

$R \neq C$:- Non Square matrix (Rectangular matrix).
 unbalanced system

(A) $R < C$ (equation < variable)
 - under determined system

$$A_{2 \times 3} = \begin{bmatrix} 1 & 2 & 4 \\ 7 & 8 & 6 \end{bmatrix}_{2 \times 3}$$

↳ Horizontal matrix

(B) $R > C$ (equation > variable)
 - over determined system

$$A_{3 \times 2} = \begin{bmatrix} 1 & 7 \\ 2 & 8 \\ 4 & 6 \end{bmatrix}_{3 \times 2}$$

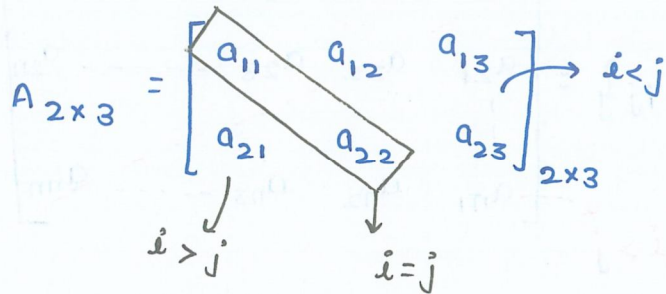
↳ Vertical matrix

(C) $R = 1, C > 1$
 $A_{1 \times 3} = [a_{11} \ a_{12} \ a_{13}] \rightarrow$ Row vector.

(D) $R > 1, C = 1$
 $A_{3 \times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \rightarrow$ Column vector.

Formation of matrix A/c to position of elements (i, j):

$$A_{2 \times 3} = \{a_{ij}\}_{2 \times 3} = \left\{ \begin{array}{ll} i+j & i > j \\ 0 & i = j \\ i-j & i < j \end{array} \right\}$$



$$A = \begin{bmatrix} 0 & -1 & -2 \\ 3 & 0 & -1 \end{bmatrix}_{2 \times 3}$$

GATE

$A_{n \times n} = \{a_{ij}\}_{n \times n} = \{i^2 - j^2 \forall i, j\}$ ^{for every} then $\sum a_{ij}$ is \dots

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{l} -ve \\ +ve \end{array} = \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}$$

Question For $A_{n \times n} = \{a_{ij}\}_{n \times n}$ Find sum of all elements for

(1) $A = \{a_{ij}\} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right\}$

* (2) $A = \{a_{ij}\} = \left\{ \begin{array}{ll} i & i = j \\ 0 & i \neq j \end{array} \right\}$

(3) $A = \{a_{ij}\} = \{1 \forall i, j\}$

* (4) $A = \{a_{ij}\} = \left\{ \begin{array}{ll} i+j & i = j \\ 0 & i \neq j \end{array} \right\}$

* (5) $A = \{a_{ij}\} = \left\{ \begin{array}{ll} i \times j & i = j \\ 0 & i \neq j \end{array} \right\}$

* (6) $A = \{a_{ij}\} = \left\{ \begin{array}{ll} i^3 & i = j \\ 0 & i \neq j \end{array} \right\}$

$$(7) A = \{i+j \quad \forall i, j\}$$

$$(8) A = \{a_{ij}\} = \{i^n - j^n \quad \forall i, j\}$$

$$A_{n \times n} = \{a_{ij}\} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

$i < j$ (top right), $i > j$ (bottom left), $i = j$ (diagonal)

Solⁿ (1)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{n \times n}$$

$$\sum a_{ij} = 1 + 1 + 1 + \dots + n \text{ times} = n$$

Solⁿ (2)

$$A = \begin{bmatrix} i=1 & 0 & 0 \\ 0 & i=2 & 0 \\ 0 & 0 & i=3 \end{bmatrix}_{n \times n}$$

$$\sum a_{ij} = 1 + 2 + 3 + \dots + n \text{ times}$$

$$= \frac{n(n+1)}{2}$$

A.P.,

$$S_n = \frac{n}{2}(a+l)$$

Solⁿ (3)

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}_{n \times n} \left. \begin{array}{l} \rightarrow \sum R_1 \\ \rightarrow \sum R_2 \\ \rightarrow \sum R_n \end{array} \right\} = \sum R_1 + \sum R_2 + \sum R_n$$

$$\sum R_1 = \sum R_2 = \sum R_n = n$$

$$\text{Total Sum} = n + n + n + \dots + n \text{ times} = n \times n = n^2$$

Solⁿ (4)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \ddots \\ & & & 2n \end{bmatrix}_{n \times n}$$

$$\sum a_{ij} = 2(1+2+3+\dots+n) = 2 \times \frac{n}{2}(1+n) = n(n+1)$$

Solⁿ (5)

$$A = \begin{bmatrix} 1^2 & & \\ & 2^2 & \\ & & \ddots \\ & & & n^2 \end{bmatrix}$$

$$\begin{aligned} \sum a_{ij} &= 1^2 + 2^2 + \dots + n^2 \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Solⁿ (6)

$$A = \begin{bmatrix} 1^3 & & \\ & 2^3 & \\ & & \ddots \\ & & & n^3 \end{bmatrix}$$

$$\begin{aligned} \sum a_{ij} &= 1^3 + 2^3 + \dots + n^3 \\ &= \left[\frac{n(n+1)}{2} \right]^2 \end{aligned}$$

Solⁿ (7)

$$A = \begin{bmatrix} 2 & 3 & 4 & \dots & (1+n) \\ 3 & 4 & 5 & \dots & (2+n) \\ \vdots & & & & \\ (n+1) & (n+2) & (n+3) & \dots & (n+n) \end{bmatrix}_{n \times n}$$

$\rightarrow \sum R_1 = (n+3) \frac{n}{2}$
 $\rightarrow \sum R_2 = \frac{n}{2} (5+n)$
 $\rightarrow \sum R_n = \frac{n}{2} [(2n+1)+n]$

$$\sum R_1 = 2 + 3 + 4 + \dots + (1+n) = \frac{n}{2} (3+n)$$

A.P Series.

$$\begin{aligned} \text{Total sum} = \sum a_{ij} &= \frac{n}{2} [(3+n) + (5+n) + (7+n) + \dots + [(2n+1)+n]] \\ &= \frac{n}{2} * \left[\frac{n}{2} \{ (3+n) + 2n+1+n \} \right] \\ &= \frac{n^2}{4} \{ (4+4n) \} = n^2 (1+n) \text{ Ans.} \end{aligned}$$

Elementary Transformation :-

- To make the element zero we use elementary transformation.

3 types of elementary transformation :-

(1) we can interchange 2 rows or columns.

$$\text{ex:- } R_1 \longleftrightarrow R_3$$

(2) we can multiply any non-zero no. to Any Line.

$$\text{ex:- } c_2 \rightarrow \frac{1}{2} c_2$$

(3) we can multiply any non-zero no. to Any Line, and Add or subtract with the another parallel Line.

$$\text{ex:- } R_3 \rightarrow R_3 - 2R_1$$

$$\begin{array}{c} \boxed{R_2} \rightarrow 2R_1 + \boxed{R_2} \\ \text{same. (det. not changed-)} \end{array}$$

Q.

$$\{a_{ij}\} = A = \begin{bmatrix} 2 & -3 & 4 \\ 5 & -8 & 1 \\ 7 & 9 & -1 \end{bmatrix}_{3 \times 3}$$

Follow the E-Ts and find

(i) $R_1 \rightarrow R_1 + R_2$

(ii) $R_3 \rightarrow R_3 + 3R_1$

and find $\sum a_{22} + a_{11}$ is ...

$$A = \begin{bmatrix} 7 & -11 & 5 \\ 5 & -8 & 1 \\ 13 & 0 & 11 \end{bmatrix}$$

$$\sum a_{22} + a_{11} = -8 + 7 = -1$$

BASIC OPERATION OF MATRIX

(1) Transpose :-

- It is valid for both square and non-square matrix.
- We can get the transpose by interchanging rows and column.
- It is represented by A^T or A' .

ex:-

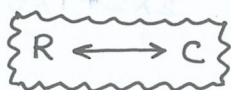
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix}_{3 \times 3}$$

$$A^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 2 \\ -2 & 1 & 4 \end{bmatrix}_{3 \times 3}$$

Interchanging



$$\{a_{ij}\}_{R \times C} \xrightarrow{T} \{a_{ji}\}_{C \times R}$$

- In diagonal element (leading) no change in Transpose of square matrix.

** Sum of elements of leading diagonal is known as Trace (tr).

$$* \boxed{tr(A) = tr(A^T)}$$

$$- \text{Trace}(tr) = \sum a_{ij}, i=j$$

(2) Matrix multiplication :-

- It is valid for both square as well as non-square matrix.

$$\begin{matrix} [A]_{x \times y} & [B]_{y \times z} & = & [A B]_{x \times z} \\ \uparrow & \uparrow & & \uparrow \\ \text{Row of 1st} & \text{Same} & \text{Col. of 2nd} & \text{Outer order of Product} \end{matrix}$$

* Inner order of Product

Example :-

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}_{2 \times 2} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$$

$$C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$$

$$[AB]_{2 \times 3} = \begin{bmatrix} 2 \times 1 + 4 \times 1 & 2 \times 2 + 1 \times 5 & 2 \times 3 + 1 \times 6 \\ 3 \times 1 + 2 \times 4 & 3 \times 2 + 2 \times 5 & 3 \times 3 + 2 \times 6 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 12 \\ 11 & 16 & 21 \end{bmatrix}_{2 \times 3}$$

$m = 12$ $Add = 6$

$$[BC]_{2 \times 2} = \begin{bmatrix} 1 \times 1 + 2 \times 2 + 3 \times 3 & 1 \times 4 + 2 \times 5 + 3 \times 6 \\ 4 \times 1 + 5 \times 2 + 6 \times 3 & 4 \times 4 + 5 \times 5 + 6 \times 6 \end{bmatrix} = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}_{2 \times 2}$$

$m = 12$ $Add = 8$

Key points :-

(i) For $A_{x \times y} \cdot B_{y \times z} = [A B]_{x \times z}$

- $x \times z$ is referred as outer order and y is referred as inner order.
- NO. of elements = outer order ($x \times z$)
(e)
 $e = x \times z$